Contraction stability and transverse stability of synchronization in complex networks

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We consider discrete dynamical networks, and analytically demonstrate the relation between transverse stability in the Milnor sense and contraction stability, the stability for synchronous manifolds obtained via the partial contraction principle. By contraction for a system, we mean that initial conditions or temporary disturbances are forgotten exponentially fast, so that all trajectories of this system converge to a unique trajectory. In addition, synchronization of star-shaped complex networks is investigated via the partial contraction principle. This example further verifies the interrelation between contraction and transverse stability.

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I. INTRODUCTION

Many evolving phenomena in nature or society may be considered as dynamical processes on networks. Examples such as the World Wide web (WWW) [1], transmission of diseases in populations [2–4], and interactions of individuals in society [5–7] have been widely and thoroughly studied. Generally, these networks can be described by coupled differential equations or coupled maps. Among the various dynamical behaviors of networks, the synchronization of the time evolution of individual nodes has been thoroughly studied [8–10]. Synchronization is observed to be important in many areas of application, from brain function and epilepsy to the emergence of coherent phenomena. The results we present in this paper offer a technique to help understand the generic conditions under which synchronization can occur.

In the past decades, many methods have been introduced to investigate the synchronization of networks. The most notable of these are the use of Lyapunov exponents [11] and the master stability function developed in [12]. But, as most methods to study synchronization start by linearizing the dynamical networks onto their synchronous manifolds, the results are just valid locally, leading only to constraints for the initial states of the nodes in the networks.

The so-called contraction principle first appeared in [13]. Basically, a nonlinear dynamical system is called "contracting" if initial conditions or temporary disturbances are forgotten exponentially fast, so that all trajectories of this system converge to a unique trajectory. Although we know it exists, the properties of this unique trajectory are completely unknown to us. In view of this fact, the contraction principle is not sufficient for the stability analysis of synchronous manifolds. Fortunately, the partial contraction principle [14] has extended the contraction principle to include convergence to behaviors or to specific properties (such as convergence to a manifold). This makes the partial contraction principle a very general analysis tool to study the stability of synchronization for dynamical systems.

Recently, there has been some research [15–17] on synchronization of dynamical networks in continuous time by using the contraction principle. Intuitively, these results imply that the conditions for synchronization obtained by the contraction principle would be stronger than those obtained by using the transverse stability in the Milnor sense, i.e., all the Lyapunov exponents should be negative. This paper analytically investigates the true reason for this by considering a class of discrete-time dynamical networks for both nonlinear and linear coupling. The analytical results verify that contraction stability is stronger than transverse stability in the Milnor sense. On the other hand, for discrete-time dynamical networks, the contraction principle has not been applied to solve the problems of synchronization. In view of this, we conclude by giving a particular example to show how to investigate synchronization for discrete-time networks via the partial contraction principle.

In Sec. II, the contraction principle and partial contraction principle are briefly reviewed with some further analysis and comments. In Sec. III, the relation between contraction stability and transverse stability of synchronization of complex networks is presented for both nonlinear and linear coupling. Finally, in Sec. IV, we take the star-shaped complex network as a particular example and show how to gain the range of coupling strength required for complete synchronization via the partial contraction principle, and thereby further verify the results obtained in Sec. III.

II. NONLINEAR CONTRACTION PRINCIPLE

Now we give a brief introduction to the contraction and partial contraction principles, whose details can be found in [13] and [14].

A discrete-time nonlinear system is given by

$$x(t+1) = F(x(t),t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $t \in \mathbb{N}$, and F is a nonlinear map. The associated virtual dynamics of system (1) is given by

$$\delta x(t+1) = \frac{\partial F(x(t),t)}{\partial x(t)} \delta x(t).$$

Then the virtual length dynamics is represented by

$$\delta x(t+1)^T \delta x(t+1) = \delta x(t)^T \frac{\partial F^T}{\partial x(t)} \frac{\partial F}{\partial x(t)} \delta x(t).$$

Thus, exponential convergence to a unique trajectory is guaranteed for

$$\frac{\partial F^T}{\partial x(t)}\frac{\partial F}{\partial x(t)} - \mathbf{I} < 0, \quad \forall \quad x, \ t \geq t_0$$

Let $\|\cdot\|_2$ denote the spectral norm of a matrix; then the above inequality becomes

$$\left\|\frac{\partial F}{\partial x}\right\|_2 < 1, \quad \forall \quad x, \ t \geq t_0.$$

The discrete-time version of the contraction principle is presented in the following. A discrete-time system given by (1) is contracting if and only if the largest singular value of the Jacobian $\partial F/\partial x$ remains smaller than 1 uniformly in discrete time: $\|\partial F/\partial x\|_2 < 1$, $\forall$ x, $t \geq t_0$. If this is the case, then all trajectories will converge exponentially to a single particular trajectory.

Suppose $\mathbf{P}$ is an invertible matrix whose order is the same as $\partial F/\partial x$. The generalized definition is as follows. The system (1) is said to be contracting if and only if $\|\mathbf{P}^{-1}(\partial F/\partial x)\mathbf{P}\|_2 < 1$, $\forall$ x, $t \geq t_0$. Then all trajectories converge exponentially to a single particular trajectory.

We now describe the *partial* contraction principle, which is based on the contraction principle described above. Consider a nonlinear system of the form

$$x(t+1) = F(x(t), x(t), t) \tag{2}$$

and assume that the auxiliary system

$$y(t+1) = F(y(t), x(t), t) \tag{3}$$

is contracting with respect to y. If a particular solution of the auxiliary y system satisfies a specific smooth property, then all trajectories of the original x system satisfy this property exponentially. Under this condition the original system is said to be partially contracting.

If the auxiliary system (3) is contracting with respect to y and $y_1 = y_2 = \cdots = y_n = y_\infty$ is the solution of that system, i.e., this is a particular trajectory, then by the contraction principle we know that every solution $y(t)$ converges to $y_\infty(t)$ exponentially. Obviously, the solution $x(t)$ of the x system (2) is also a particular solution of the y system (3). Then we get that $y(t)$ converges to $x(t)$ exponentially.

III. RELATIONSHIP BETWEEN CONTRACTION AND TRANSVERSE STABILITY OF SYNCHRONIZATION

In this paper, we consider a network of the form

$$x_i(t+1) = f(x_i(t)) + \sum_{j \in \mathcal{N}_i} k_{ij}[g(x_j(t)) - g(x_i(t))],$$

where $i = 1, \ldots, n$, $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, and $f: \mathbb{R} \to \mathbb{R}$ is a continuously differential map defining the local nonlinear map. $g: \mathbb{R} \to \mathbb{R}$ is also a continuously differential map, i.e., the coupling function. We refer to the coupling function $g(x) = f(x)$ as nonlinear coupling and $g(x) = x$ as linear coupling. $\mathcal{N}_i$ denotes the set of indices of the active links of element i. The number of indices of $\mathcal{N}_i$ is denoted by k_i, i.e., the degree of node i, and $k_{ij} = \varepsilon / k_i$, $\forall$ $j \in \mathcal{N}_i$. The constant ε represents the coupling strength, restricted to $0 < \varepsilon < 1$. From above, we know that $\Sigma_{j \in \mathcal{N}_i} k_{ij} = \varepsilon$ and $k_{ii} = -\varepsilon$.

The n nodes are said to achieve complete synchronization if

$$\lim_{t \to \infty} \|x_i(t) - s(t)\| = 0, \quad i = 1, 2, \ldots, n,$$

where $s(t) \in \mathbb{R}$ is a solution of an isolated node, namely, $s(t+1) = f(s(t))$.

Next, we will show the relation between contraction stability and transverse stability in the Milnor sense for both nonlinear and linear coupling.

A. Nonlinear coupling

First, we consider dynamical networks with nonlinear coupling. In this case, the networks can be described by

$$x_i(t+1) = (1-\varepsilon) f(x_i(t)) + \sum_{j \in \mathcal{N}_i} k_{ij} f(x_j(t)). \tag{4}$$

We remark here that Eq. (4) is a special case of nonlinear coupling, and in this paper we will perform analysis only for this special case.

Let $\mathbf{L}$ be the Laplacian matrix of the network, i.e., $\mathbf{L} = \Sigma_{i,j \in \mathcal{N}} \mathbf{L}_{ij}$, $\mathcal{N} = \cup_{i=1}^{n} \mathcal{N}_i$, and

$$\mathbf{L}_{ij} = \begin{pmatrix} \ddots & \vdots & & \vdots & \\ \cdots & k_{ij} & \cdots & -k_{ij} & \cdots \\ & \cdots & \ddots & \vdots & \\ \cdots & -k_{ji} & \cdots & k_{ji} & \cdots \\ & \vdots & & \vdots & \ddots \end{pmatrix}_{n \times n},$$

where all the elements in $\mathbf{L}_{ij}$ are zero except those displayed above at the four intersections of the ith and jth rows with the ith and jth columns. Let

$$F(x(t)) = (f(x_1), f(x_2), \ldots, f(x_n))^T.$$

Equation (4) can then be rewritten in matrix form:

$$x(t+1) = (\mathbf{I} - \mathbf{L}) F(x(t)), \tag{5}$$

where $\mathbf{I}$ is an $n \times n$ unit matrix.

Construct an auxiliary system of system (5) as

$$y_i(t+1) = f(y_i(t)) + \sum_{j \in \mathcal{N}_i} k_{ij}[f(y_j(t)) - f(y_i(t))] - \alpha \sum_{j=1}^{n} f(y_j(t)) + \alpha \sum_{j=1}^{n} f(x_j(t)), \tag{6}$$

which has a particular solution $y_1 = y_2 = \cdots = y_n = y_\infty$ with $y_\infty(t+1) = f(y_\infty(t)) - n\alpha f(y_\infty(t)) + \alpha \Sigma_{j=1}^{n} f(x_j(t))$, where $\alpha \in \mathbb{R}$ is an undetermined coefficient.

By the partial contraction principle in Sec. III, if the auxiliary system (6) is contracting with respect to y, then all trajectories of system (4) will satisfy the independent property $x_1 = x_2 = \cdots = x_n$. Note also that $y_\infty(t)$ will converge exponentially to $s(t)$, which is a solution of $s(t+1) = f(s(t))$, by the analysis in Sec. II.

Let $\tilde{\mathbf{J}}=\mathbf{I}-\mathbf{L}-\alpha\mathbf{T}$; then the Jacobian matrix of system (6) is given by $\mathbf{J}=\tilde{\mathbf{J}}\mathbf{D}(y(t))$, where

$$\mathbf{D}(y(t)) = \frac{\partial F(y)}{\partial y} = \begin{pmatrix} f'(y_1) & & & \\ & f'(y_2) & & \\ & & \ddots & \\ & & & f'(y_n) \end{pmatrix}$$

and

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n}.$$

Obviously, 1 is an eigenvalue of the matrix $\mathbf{I}-\mathbf{L}$. Let 1 and $\lambda_2,\ldots,\lambda_n$ be all the eigenvalues of $\mathbf{I}-\mathbf{L}$; then it is easy to see that $\lambda_1=1-n\alpha,\lambda_2,\ldots,\lambda_n$ are all the eigenvalues of $\tilde{\mathbf{J}}$.

By the contraction principle, if system (6) is contracting with respect to y, then we have

$$\|\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{D}(y(t))\mathbf{P}\|_2 < 1, \quad \forall \quad t \geq t_0, \tag{7}$$

for an invertible matrix $\mathbf{P}$; here t_0 is a sufficiently larger positive integer. Let $z(t)=(s(t),s(t),\ldots,s(t))^T$, and due to the continuity of f', condition (7) leads to

$$\|\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{D}(z(t))\mathbf{P}\|_2 < 1, \qquad \forall\, t \geq t_0. \tag{8}$$

So, for every $m\in\{1,2,\ldots\}$, we get

$$\prod_{i=1}^{m} \|\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{D}(z(t_0+i))\mathbf{P}\|_2 < 1. \tag{9}$$

This means that

$$\|\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{P}\|_2^m \prod_{i=1}^{m} |f'(s(t_0+i))| < 1. \tag{10}$$

Using matrix identities [18,19], there must exist an invertible matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{P} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix};$$

then from (10) we have

$$[\max_{i=1}^{n}\{|\lambda_i|\}]^m \prod_{j=1}^{m} |f'(s(t_0+j))| < 1.$$

So we can obtain that

$$|\lambda_i|^m \prod_{j=1}^{m} |f'(s(t_0+j))| < 1, \quad i=1,2,\ldots,n. \tag{11}$$

Since we can always choose α such that $|\lambda_1|=|1-n\alpha|=\min_{i=1}^{n}\{|\lambda_i|\}$, the inequalities (11) give

$$\frac{1}{m}\sum_{t=1}^{m} \ln|f'(s(t+t_0))| + \ln|\lambda_i| < 0, \quad i=2,3,\ldots,n. \tag{12}$$

Thus, we can obtain

$$\lim_{m\to\infty} \frac{1}{m}\sum_{t=1}^{m} \ln|f'(s(t+t_0))| + \ln|\lambda_i| < 0, \tag{13}$$

for every $i\in\{2,3,\ldots,n\}$. Now, defining $\tilde{s}(t)=s(t+t_0)$, we can deduce that $\tilde{s}(t+1)=f(\tilde{s}(t))$, $t=1,2,\ldots$, leading to $f'(s(t+t_0))=f'(\tilde{s}(t))$. Inequality (13) becomes

$$\lim_{m\to\infty} \frac{1}{m}\sum_{t=1}^{m} \ln|f'(\tilde{s}(t))| + \ln|\lambda_i| < 0,$$

that is,

$$\Lambda + \ln|\lambda_i| < 0, \quad i=2,3,\ldots,n. \tag{14}$$

Here Λ is the Lyapunov exponent of f.

Let $\Lambda_i=\Lambda+\ln|\lambda_i|$. It is easy to see $\{\Lambda_i\}_{i=2}^{n}$ are just the $n-1$ transverse Lyapunov exponents for system (4). The inequality group (14) is just the condition for transverse stability of the synchronous manifold in the Milnor sense, i.e., the stability is for almost all or at least a positive measure set of initial conditions. At the same time, note that generally the processes of deduction from (12) to (13) and from (7) to (8) are all irreversible, unless some constraints are imposed on f and the coupling scheme in system (4).

B. Linear coupling

The dynamical networks with linear coupling can be described as

$$x_i(t+1) = f(x_i(t)) + \sum_{j\in\mathcal{N}_i} k_{ij}(x_j(t)-x_i(t)). \tag{15}$$

Equation (15) can be rewritten in matrix form as

$$x(t+1) = F(x(t)) - \mathbf{L}x, \tag{16}$$

where the matrix $\mathbf{L}$ and map F have the same definitions as in Sec. III A.

Construct an auxiliary system of system (16) as

$$y_i(t+1) = f(y_i(t)) + \sum_{j\in\mathcal{N}_i} k_{ij}[y_j(t)-y_i(t)] - \alpha\sum_{j=1}^{n} y_j(t) + \alpha\sum_{j=1}^{v} x_j(t), \tag{17}$$

which also has a particular solution $y_1=y_2=\cdots=y_n=y_\infty$ with $y_\infty(t+1)=f(y_\infty(t))-n\alpha y_\infty(t)+\alpha\Sigma_{j=1}^{n}x_j(t)$.

The Jacobian matrix of system (17) with respect to y is given by

$$\mathbf{J} = \frac{\partial F}{\partial y(t)} - \mathbf{L} - \alpha\mathbf{T}.$$

By the contraction principle, if system (17) is contracting with respect to y, then we have

$$\left\| \mathbf{P}^{-1}\left(\frac{\partial F}{\partial y(t)} - \mathbf{L} - \alpha \mathbf{T}\right)\mathbf{P} \right\|_2 < 1, \quad \forall \quad t \geq t_0, \tag{18}$$

for an invertible matrix $\mathbf{P}$. Due to the continuity of f' and with adequately large time t_0, condition (18) becomes

$$\left\| \mathbf{P}^{-1}\left(\frac{\partial F}{\partial z(t)} - \mathbf{L} - \alpha \mathbf{T}\right)\mathbf{P} \right\|_2 < 1, \quad \forall \quad t \geq t_0. \tag{19}$$

For every $m \in \{1,2,\ldots\}$, we get

$$\prod_{i=1}^{m} \left\| \mathbf{P}^{-1}\left(\frac{\partial F}{\partial z(t_0+i)} - \mathbf{L} - \alpha \mathbf{T}\right)\mathbf{P} \right\|_2 < 1. \tag{20}$$

Because $-\mathbf{L}$ can be diagonalized and its eigenvalues are denoted by 0 and λ_i $(i=2,\ldots,n)$, then there must exist an invertible matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\left(\frac{\partial F}{\partial z(t_0+i)} - \mathbf{L} - \alpha \mathbf{T}\right)\mathbf{P} = \begin{pmatrix} f'(s(t_0+i)) - n\alpha & & & \\ & f'(s(t_0+i)) + \lambda_2 & & \\ & & \ddots & \\ & & & f'(s(t_0+i)) + \lambda_n \end{pmatrix}. \tag{21}$$

Noting the equality (21), then from (20) we have

$$\prod_{j=1}^{m} \max_{1 \leq i \leq n} |f'(s(t_0+j)) + \lambda_i| < 1. \tag{22}$$

So from (22) we obtain that

$$\prod_{j=1}^{m} |f'(s(t_0+j)) + \lambda_i| < 1, \quad i = 1,2,\ldots,n. \tag{23}$$

Since we can always choose α such that $|f'(s(t_0+i)) - n\alpha| < 1$, inequality (23) becomes

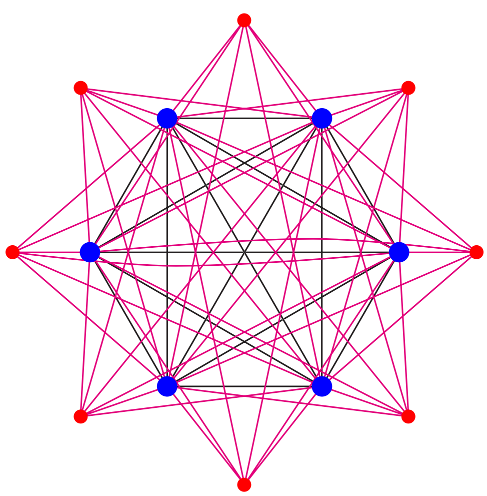

FIG. 1. (Color online) Star-shaped network with $n=14$, $k=6$. The big (blue) dots denote the center nodes, and the remaining small (red) dots the noncenter nodes.

$$\lim_{m\to\infty} \frac{1}{m} \sum_{t=1}^{m} \ln|f'(\tilde{s}(t)) + \lambda_i| < 0, \quad i = 2,\ldots,n, \tag{24}$$

where $\tilde{s}(t)$ has the same definition as in Sec. III A.

It is easy to see that these $n-1$ inequalities in (24) are just the conditions for transverse stability of the synchronous manifold in the Milnor sense. And, in general, the processes of deduction from (23) to (24) and from (18) to (19) are irreversible.

As mentioned in the Introduction, the contracting property of a dynamical network can guarantee that all its trajectories converge to a unique trajectory. On the other hand, the transverse stability of a synchronous manifold of this network ensures that many trajectories will converge to this manifold. So we can state that a combination of the contracting property of a dynamical network and transverse stability of its synchronization manifold can guarantee the global stability of that manifold.

IV. SYNCHRONIZATION IN STAR-SHAPED COMPLEX NETWORKS

In this section, we take multicenter complex networks with nonlinear coupling as examples to illustrate the use of the contraction principle in complete synchronization, and make an explicit contrast between contraction stability and transverse stability. The continuously differential map in (4) is defined as $f(x)=ax(1-x)$, $x \in [0,1]$ and $a=4$, i.e., the chaotic logistic map. For simplicity, we consider multicenter networks with k centers; all center nodes are globally coupled, and all noncenter nodes connect to every center but have no direct connections among themselves [20]. In system (4), let indices from 1 to k denote the center nodes, and

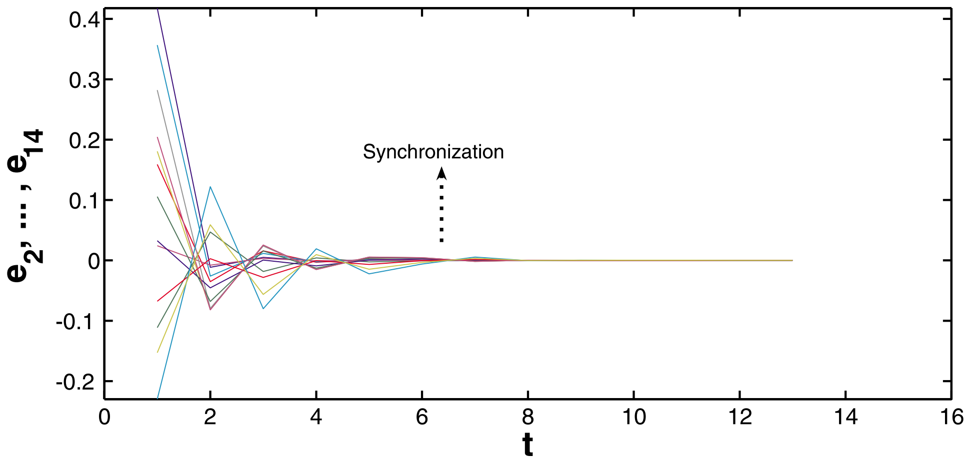

FIG. 2. (Color online) Synchronization errors with $\varepsilon=0.755 \in I_c$.

the remaining $n-k$ indices denote noncenter nodes. In this case, we have

$$\mathbf{I}-\mathbf{L}=\begin{bmatrix} 1-\varepsilon & \frac{\varepsilon}{n-1} & \cdots & \frac{\varepsilon}{n-1} & \cdots & \cdots & \frac{\varepsilon}{n-1} \\ \frac{\varepsilon}{n-1} & 1-\varepsilon & \cdots & \frac{\varepsilon}{n-1} & \cdots & \cdots & \frac{\varepsilon}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \cdots & \vdots \\ \frac{\varepsilon}{n-1} & \frac{\varepsilon}{n-1} & \cdots & 1-\varepsilon & \cdots & \cdots & \frac{\varepsilon}{n-1} \\ \frac{\varepsilon}{k} & \frac{\varepsilon}{k} & \cdots & \frac{\varepsilon}{k} & 1-\varepsilon & & \\ \vdots & \vdots & \cdots & \vdots & & \ddots & \\ \frac{\varepsilon}{k} & \frac{\varepsilon}{k} & \cdots & \frac{\varepsilon}{k} & & & 1-\varepsilon \end{bmatrix}$$

and then the eigenvalues of $\tilde{\mathbf{J}}$ are given by

$$\lambda_1 = 1 - n\alpha, \quad \lambda_3 = \cdots = \lambda_{k+1} = 1 - \frac{n}{n-1}\varepsilon,$$

$$\lambda_2 = 1 - \frac{2n-k-1}{n-1}\varepsilon, \quad \lambda_{k+2} = \cdots = \lambda_n = 1-\varepsilon.$$

By noting the inequality (8) and the analysis at the end of Sec. III, this star-shaped network can achieve complete synchronization globally, if there exists an invertible matrix $\mathbf{P}$ such that

$$\|\mathbf{P}^{-1}\tilde{\mathbf{J}}\mathbf{P}\|_2 < \frac{1}{a}, \tag{25}$$

since $|f'| \leq a$ and $\mathbf{D}(z(t)) = f'\mathbf{I}$. Now let $n=14$, $k=6$, and $a=4$; the configuration of this star-shaped network is shown by Fig. 1. By the analysis in Sec. III, condition (15) can be satisfied if

$$\max_{i=2}^{14} |\lambda_i| < \frac{1}{4}. \tag{26}$$

Solving (16), we obtain $\varepsilon \in \left(\frac{3}{4}, \frac{65}{84}\right) \triangleq I_c$ with which synchronization occurs. Then it is easy to see that $\varepsilon \in \left(\frac{1}{2}, \frac{39}{42}\right) \triangleq I_t$ is needed for transverse stability. Note that $I_c \subset I_t$, which agrees with the relationship derived in Sec. III between contraction and transverse stability. Set $e_i(t) = x_1(t) - x_i(t)$ for $i \in \{2,3,\ldots,14\}$, that is, $e_i(t)$ are the synchronization errors. The synchronization errors under coupling strengths $\varepsilon = 0.755 \in I_c$ and $\varepsilon = 0.501 \in I_t \backslash I_c$ are plotted in Figs. 2 and 3, respectively. The chaotic motion of node x_1 is shown by Fig.

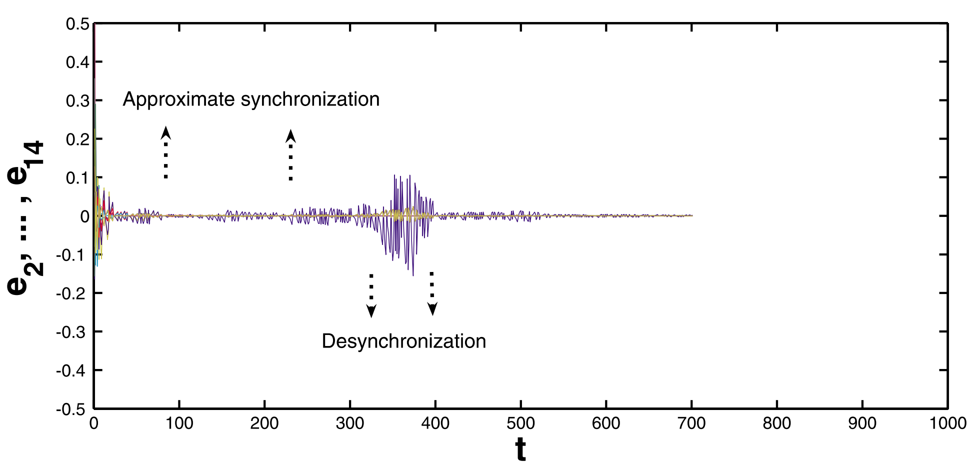

FIG. 3. (Color online) Synchronization errors with $\varepsilon=0.501 \in I_t$.

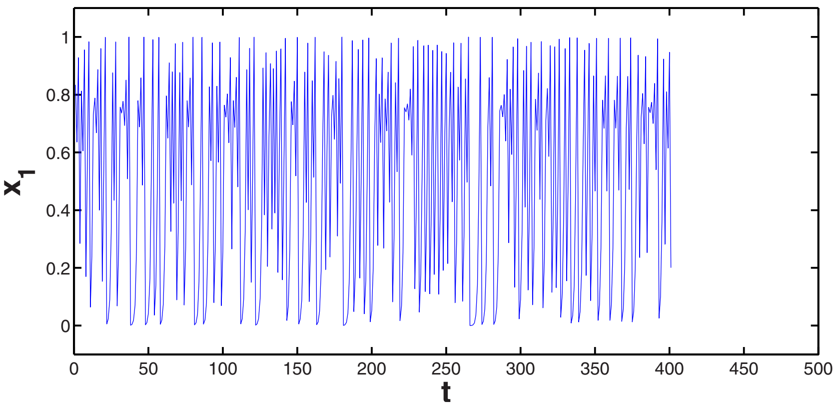

FIG. 4. (Color online) Chaotic motion of node $x_1(t)$ with $\varepsilon = 0.755 \in I_c$.

4. Figure 2 illustrates that this network can achieve complete synchronization globally and quickly relative to the time ranges shown in Fig. 2. In Fig. 3, we can see a transition process from approximate synchronization to desynchronization. Since this transition is observed during a large time interval (100–400), whether this system can achieve synchronization is unknown to us. Even this system appears to synchronize for larger times; thus we can affirm that the speed of synchronizing is slow. From above, we can conclude that contraction stability is global and the region of coupling strength guaranteeing contraction stability could result in faster speed for synchronization than the region assuring transverse stability.

V. CONCLUSIONS AND REMARKS

By considering a class of discrete dynamical networks for both nonlinear and linear coupling, we have analytically verified some intuitional results, that is, contraction stability is stronger than transverse stability in the Milnor sense. In this paper, we perform an analysis for nonlinear coupling only in the special case Eq. (4). For other classes of dynamical networks, there may very likely exist the same interrelation between the two stabilities, which could be verified by a method similar to that presented here. To do this, suitable auxiliary systems corresponding to those networks need to be successfully constructed.

In fact, this interesting relation is quite natural. The contraction principle takes a global view of the motions of dynamical systems, and it requires the persistent decrease of distance between every two trajectories after a certain time t_0. On the other hand, transverse stability relates to linearization on a synchronous manifold; therefore results obtained by transverse stability analysis are valid only locally. Transverse stability does not require the persistent decrease of distance between every two trajectories, but just a decrease on average. Because of this relationship, the region of coupling strength for complete synchronization obtained by transverse stability is larger than that gained by contraction stability, as we see in Sec. IV. At the same time, we must note that contraction stability may not be a necessary but only a sufficient condition for global synchronization. We are currently exploring this issue. For discrete-time dynamical networks, the method developed in this paper for synchronization can also be applied to networks with various configurations or to other modes of synchronization. Considering the relations $\|\cdot\|_2 \leq \sqrt{n}\|\cdot\|_\infty$ and noting the inequality (8), it is possible to obtain the regions of coupling strength for synchronization without computing the eigenvalues of the coupling matrix, which may often be demanding. Our further research will concentrate on the above problems.

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